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Practice Midterm 2 May 7, 2009

1. Show that there is a unique continuous function on $[0,1]$ which solves the functional equation

$$\mu(x) = \frac{2}{3}\mu(1-x^3) + \frac{\sin(\pi x)}{3} \quad \forall x \in [0,1]$$

(Hint: Contraction mapping!)

Solution. The mapping T defined by

$$Tf(x) = \frac{2}{3}\mu(1-x^3) + \frac{\sin(\pi x)}{3}$$

clearly maps $C([0,1])$ to $C([0,1])$. Further, it is a contraction since

$$\sup_{x\in[0,1]} |Tf(x) - Tg(x)| = \frac{2}{3}\sup_{x\in[0,1]} |f(x) - g(x)|$$

Since $C([0,1])$ is complete in the sup norm, the contraction mapping theorem gives the existence of a unique fixed point for the map T in $C([0,1])$, that is there is a unique continuous function μ solving $T\mu = \mu$.

2. Let $\phi : \mathbb{R} \to \mathbb{R}^2$ be the function $x \mapsto \frac{1}{1+x^2}(2x, 1-x^2)$. An easy calculation shows that

$$d(x,y) \equiv ||\phi(x) - \phi(y)||_2 = \frac{2|x-y|}{\sqrt{(1+x^2)(1+y^2)}}.$$

Show that $(\mathbb{Q}, d)$ is a metric space, and find (with justification) the completion of $\mathbb{Q}$ with respect to this metric. (Hint: What is $||\phi(x)||_2$?)

Solution. From $d(x,y) = 2|x-y|/\sqrt{(1+x^2)(1+y^2)}$, it is easy to show that $d(x,y) \geq 0$ $\forall x,y$ and $d(x,y) = 0 \implies x = y$. Also, $d(x,y) = ||\phi(x) - \phi(y)||_2$, d satisfies the triangle inequality. Thus, $(\mathbb{Q}, d)$ is indeed a metric space.

A direct computation shows that $||\phi(x)||_2 = 1$ for all $x \in \mathbb{Q}$, so that $\phi : (\mathbb{Q}, d) \to (S^1, ||\cdot||_2)$ is an isometry, where $S^1 = \left\{z \in \mathbb{R}^2 \middle| ||z||_2 = 1\right\}$ is the unit circle, which is a closed subset of $\mathbb{R}^2$ in the topology induced by l^2 norm. Since every closed subset of a complete metric space is also complete, $(S^1, ||\cdot||_2)$ is a complete metric space.

If $z \in S^1 \neq (0,-1)$, it is easy to see that there exists $x \in \mathbb{R}$ s.t. $\phi(x) = z$. By continuity of ϕ and the density of $\mathbb{Q}$ in $\mathbb{R}$, it follows that $\phi(\mathbb{Q})$ is dense in $S^1 \setminus (0,-1)$ in the topology given by the metric $||\cdot||_2$. It is also easy to see that $S^1 \setminus (0,-1)$ is dense in S^1 in $||\cdot||_2$. Alternatively $\forall n \in \mathbb{N}$, it is clear that $n \in \mathbb{Q}$ and $\phi(n) \to (0,-1)$ so that $\phi(\mathbb{Q})$ is dense in S^1.

Putting all of this together, we see that ϕ is extendable to an isometry between the completion of $(\mathbb{Q}, d)$ and $(S^1, ||\cdot||_2)$. For all $z \in S^1 \neq (0,-1)$, there is a unique $x \in \mathbb{R}$ s.t. $\phi(x) = z$. We introduce an ideal point, symbolically denoted by ∞ and extend ϕ to $\mathbb{R} \cup \infty$ by $\phi(\infty) = (0,-1)$. Then, the completion can also be identified with $\mathbb{R} \cup \{\infty\}$ with $d(x,y) = ||\phi(x) - \phi(y)||_2$ so that, for $x, y \in \mathbb{R}$ continuity of ϕ implies that

$$d(x,y) = \frac{2|x-y|}{\sqrt{(1+x^2)(1+y^2)}}.$$

If $y = \infty$, we have

$$d(x,\infty) = ||\phi(x) - (0,-1)||_2 = \frac{2}{\sqrt{1+x^2}}$$

3. $f_n : (X, \mathcal{B}) \to \mathbb{R}$ is a sequence of measurable functions. Show that

$$\{x| \lim_k \inf_{n\geq k} f_n(x) < t\} = \bigcup_{j=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\left\{x\middle|f_n(x) < t - \frac{1}{j}\right\}$$

and use this to conclude that $f(x) = \liminf_n f_n(x)$ is a measurable function.

Solution. This is a standard argument so the proof is a little sketchy.

If $\lim_k \inf_{n\geq k} f_n(y) < t$, it follows that there exists a subsequence $f_{n_m}(y)$ which converges to a limit less than t. Consequently, there is a $p \in \mathbb{N}$ and an $M \in \mathbb{N}$ s.t. for all $m \geq M$, it follows that $f_{n_k}(y) < t - 1/p$. Since $n_m \to \infty$, it follows that, for all $k \in \mathbb{N}$, there is an index m s.t. $n_m \geq k$, i.e.

$$y \in \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\left\{x\middle|f(x) < t - \frac{1}{p}\right\}.$$

Since $p \in \mathbb{N}$, it also follows that

$$y \in \bigcup_{j=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\left\{x\middle|f(x) < t - \frac{1}{j}\right\}.$$

thereby proving that

$$\{x| \lim_k \inf_{n\geq k} f_n(x) < t\} \subseteq \bigcup_{j=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\left\{x\middle|f(x) < t - \frac{1}{j}\right\}$$

Conversely, assume that $\lim_k \inf_{n\geq k} f_n(y) \geq t$. Then, it follows that for all $j \in \mathbb{N}$, there exists an $K(j) < \infty$ s.t. $k \geq K(j) \implies \inf_{n\geq k} f_n(y) > t - 1/j$, i.e. for all $n \geq K$, $f_n(y) > t - 1/j$. Thus, $y \notin \bigcup_{n=K(j)}^{\infty}\{x|f_n(x) < t - 1/j\}$.

In particular, this implies that $y \notin \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\{x|f_n(x) < t - 1/j\}$. Since this argument works for all $j \in \mathbb{N}$, it also follows that

$$y \notin \bigcup_{j=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\left\{x\middle|f(x) < t - \frac{1}{j}\right\}.$$

Combining this with the earlier argument proves the claim.

4.

$$A = \left\{ \mathbf{x} \in l^2(\mathbb{R}, \mathbb{N}) \middle| \sum_{i=1}^{\infty} 2^i |x_i|^2 \leq 1 \right\}.$$

Show that A is compact in the metric topology on l^2.

Solution. Note that, if $\mathbf{x} \in A$, then $|y_i| \leq 2^{-i/2}$, which also implies that, for all $\epsilon > 0$, there is an index k, *only depending on* ϵ and not on the particular $\mathbf{y}$ s.t.

$$\left[\sum_{i=k+1}^{\infty} |x_i|^2 \right]^{1/2} < \epsilon/2.$$

We can use this observation to show that for all $\epsilon > 0$, there is a finite ϵ net i.e., a finite set $S^\epsilon \subset A$ s.t. for all $\mathbf{y} \in A$, there is an index $\mathbf{x} \in S^\epsilon$ s.t. $||\mathbf{y} - \mathbf{x}|| < \epsilon$. Then you can follow the arguments from the homework to show that A is compact because A is a complete metric space with a finite ϵnet for all $\epsilon > 0$.

Let k be as above, i.e. $\mathbf{y} \in A \implies \left[\sum_{i=k+1}^{\infty} |x_i|^2 \right]^{1/2} < \epsilon/2$. Let $A' \subseteq \mathbb{R}^k$ be the set

$$A' = \{ \mathbf{x} \in \mathbb{R}^k | \sum_{i=1}^{k} 2^i |x_i|^2 \leq 1 \}.$$

A' is closed and bounded in $\mathbb{R}^k$, and is hence is compact. It follows, by considering the finite subcover of the covering $\{B_{\epsilon/2}(\mathbf{x}), \mathbf{x} \in A'\}$, that there is a finite set T^ϵ s.t. for all $\mathbf{y} \in A'$, there is an $\mathbf{x} \in T^\epsilon$ s.t. $||\mathbf{y} - \mathbf{x}|| < \epsilon/2$.

Define S^ϵ as the set of all l^2 sequences obtained by taking the elements of T^ϵ and appending zeros for all indices greater than k. If $\mathbf{y} \in A$, let $\mathbf{y}'$ denote the k-tuple of the first k elements. Then $\mathbf{y}' \in A'$ and is thus within $\epsilon/2$ of one of the elements of T^ϵ, or equivalently the initial k-tuple from one of the elements in S^ϵ. Combining this with our initial observation, we see that S^ϵ is a finite ϵ-net. Next, we argue that $(A, ||\cdot||_2)$ is a complete metric space. Since $l^2(\mathbb{R}, \mathbb{N})$ is complete, it suffices to show that A is closed. Assume that $\mathbf{x}^{(n)} \to \mathbf{x}$ in $l^2(\mathbb{R}, \mathbb{N})$. Since $||\mathbf{x}^{(n)} - \mathbf{x}||_2 \geq |x_i^{(n)} - x_i|$ for each index $i \in \mathbb{N}$, it follows that for each index i, $x_i^{(n)} \to x_i$, so that $2^i |x_i^{(n)}|^2 \to 2^i |x_i|^2$. Fatou's lemma now implies that

$$\sum_{i=1}^{\infty} 2^i |x_i|^2 = \sum_{i=1}^{\infty} \liminf_n 2^i |x_i^{(n)}|^2 \leq \liminf_n \sum_{i=1}^{\infty} 2^i |x_i^{(n)}|^2 \leq 1$$

so that $\mathbf{x} \in A$. Thus A is a closed subset of a complete metric space and is hence complete. (You can argue this directly using a Cauchy sequence in A and Fatou's lemma on the pointwise limit, exactly as we did here).

Now, we use the result from the hw.

Proposition. *(A, d) is a complete metric space and it has a finite ϵ-net for all $\epsilon > 0$. Then A is compact.*

Therefore, A is compact in the metric topology on l^2.

5. Compute the following limit or show that it doesn't exist

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(1 + \frac{k}{n^2}\right)^n$$

Carefully justify all your steps, making sure you state the names of any convergence theorems that you use.

Solution. Since we are summing many $O(1)$ terms and then dividing by the total number of terms, we should interpret these objects as "Riemann-like" sums approximating an integral. To do this, we let $k = \lceil nx \rceil$ where $0 \leq x \leq 1$. Then the sums can be recast as

$$t_n = \int_0^1 \left(1 + \frac{1}{n} \frac{\lceil nx \rceil}{n}\right)^n dx$$

Using $x \leq \frac{\lceil nx \rceil}{n} \leq x + 1/n$ and $(1 + \theta/n)^n < d^\theta$, we see that the integrand is dominated uniformly by e^x, and further it converges pointwise to e^x as $n \to \infty$. Consequently, the dominated convergence theorem implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(1 + \frac{k}{n^2}\right)^n = \int_0^1 e^x dx = e - 1.$$